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Gaussian ensemble of 2 × 2 pseudo-Hermitian random matrices

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Abstract
We present a random matrix theory for systems invariant under the joint action of parity, $\mathcal{P}$, and time reversal, $\mathcal{T}$, and, more generally, for pseudo-Hermitian systems. This brings out the appearance of the metric in a systematic way so that consistency with the postulates of quantum mechanics is maintained. Here we specialize only to 2 × 2 matrices and we construct a pseudo-unitary group. With explicit examples, nearest-neighbour level-spacing distributions for various classes of ensembles are found to exhibit a degree of level repulsion different from those hitherto known. This work is not only relevant to quantum chaos, but also to two-dimensional statistical mechanics and consistent non-local relativistic theories.

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1. Motivation

It has been generally believed that an indefinite metric will make quantum theory inconsistent. Although it has been discussed in detail that no such inconsistency arises [1], this has not become a part of conventional quantum mechanics. It has long since been known that certain divergences in physics can be overcome by the use of an indefinite metric [2]. Due to two equivalent versions of quantum electrodynamics—namely, the Gupta–Bleuler [3] form, which involves an indefinite metric, and the Dirac–Schwinger [4] form, which involves a definite metric along with an instantaneous Coulomb interaction—it is believed that a quantum theory formulated with an indefinite metric can be reformulated without an indefinite metric. It seems, thus, that the indefinite metric with local interactions may actually be a way to present a theory with a definite metric but non-local interactions. A two-level system, represented by a 2×2 Hamiltonian matrix, has been discussed by Sudarshan [1], which is pseudo-Hermitian with an indefinite metric. Also, a pseudo-unitary S-matrix was introduced to follow the time-dependent process. It was suggested that an indefinite metric could be introduced to construct a truly dynamical theory of quantized fields [5].

At about the same time, Pease developed the concept of K-Hermitian ($A^{\#} = K^{-1}A^{\dagger}K = A$) and K-unitary ($A^{\#} = A^{-1}$) matrices in the electromagnetic theory of transmission lines, discussed at length in his book [6].

Later, by introducing a metric, a general criterion for a set of non-Hermitian operators was established, consistent with the conventional interpretation of quantum mechanics [7]. A treatment of non-Hermitian operators is very useful, particularly as they may appear rather simply. For instance, let us consider a Hilbert space endowed with a scalar product between two vectors. If we perform a similarity transformation on the set of Hermitian operators, we obtain operators which are non-Hermitian with respect to the scalar product. Of course, this does not lead to any inconsistency. Sometimes, this can even be put to important use, for example concerning the boson mapping of fermion systems [8].

Without doubting the beauty of these past works, it is seen that there is no discussion about possible symmetries, as they are concentrated on questions of metric and locality–non-locality of interactions. The question is, therefore, when we encounter pseudo-Hermitian systems or non-Hermitian systems or pseudo-unitary systems, whether there is some new symmetry that we can define on the space of vectors? With such a symmetry, we could make a group. In this paper we address these issues. After this discussion, we present random matrix theory for an ensemble of pseudo-Hermitian matrices. A Gaussian distribution is chosen for convenience.

Owing to some recent works, the notion of pseudo-Hermiticity has been revived, which was discussed first by Lee [2]. It has been found that there are certain Hamiltonians describing quantum systems which possess real eigenvalues even though they are not Hermitian. Many of these systems are invariant under space–time reflection, i.e. invariant under only a joint action of parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) [9–11]. In this context, the concept of pseudo-Hermiticity was introduced [12] where it was shown that $\mathcal{PT}$-symmetry is a special case of pseudo-Hermiticity. pseudo-Hermiticity of an operator or a matrix O is simply defined through the condition: $\mathsf{O}^{\dagger} = \boldsymbol{\eta}\mathsf{O}\boldsymbol{\eta}^{-1}$ where $\boldsymbol{\eta}$ is a metric and † represents the usual adjoint or conjugate-transpose. Remarkably, it was subsequently shown that non-$\mathcal{PT}$ invariant systems that possess real eigenvalues are also pseudo-Hermitian [13]. In the above discussion, there are physical situations of great interest, including two-dimensional statistical mechanics where parity and time reversal are broken (preserving $\mathcal{PT}$) [14–16], quantum chromodynamics where chiral ensembles are used to describe the statistical properties of the lattice Dirac operator [17], spin–rotation coupling leading to an anomalous g-value for the muon [18] and related fields.

The problem of two-dimensional statistical mechanics is obviously connected with anyon physics and hence to the behaviour of an electron in an Aharonov–Bohm medium [19], i.e. a medium filled with non-quantized magnetic fluxes, reminiscent of the theory of the fractional quantum Hall effect [20]. It is important to note here that there is also another motivation which stems from a speculation by Nambu [21] that this might serve as a model for theoretical ideas such as quark confinement in a medium of monopoles. In this context, it is known that the spectral fluctuations of an Aharonov–Bohm billiard exhibit an interpolating behaviour with respect to the strength of the flux line [22]. These billiards are experimentally realized in terms of quantum dots in the presence of flux lines. It is of great interest to find an appropriate random matrix description for such $\mathcal{PT}$-invariant systems.

2. Quantum mechanics with an indefinite metric

In this section, we present the definitions of pseudo-Hermiticity and pseudo-unitarity. We illustrate these concepts with the help of examples.

2.1. *Summary of salient works*

In the past ten years, several non-Hermitian Hamiltonians have been studied, which possess real eigenvalues under certain conditions. In a detailed computational work, Bessis [23] found that all the discrete eigenvalues of the potential $V(x) = \mathrm{i}x^3$ turn out to be real. Hatano and Nelson [24] have reported that the Hamiltonians, such as $H_{H-N} = \frac{(p+\mathrm{i}g)^2}{2m} + V(x)$ with a real random potential, possess real (complex-conjugate pairs) eigenvalues if the value of an effective parameter is less (more) than a certain critical value. Certain aspects of non-Hermitian quantum mechanics have been used to discuss the density of states of disordered systems; see, for instance, [25]. Bender and Boettcher [9] have proposed a very interesting conjecture that, if a Hamiltonian is invariant under the combined transformation of $\mathcal{P}(x \to -x)$ and $\mathcal{T}$, the discrete eigenvalues will be real if the wavefunction changes only by a phase under $\mathcal{PT}$ operation; otherwise the eigenvalues will be complex-conjugate pairs (an example of spontaneous symmetry breaking). Many analytically and numerically solved examples have been proposed to support this conjecture. The eigenstates of the $\mathcal{PT}$-symmetric Hamiltonians were found to be orthogonal in a new way [11] and hence the norm was found to be indefinite [26]. Remarkably, several non-$\mathcal{PT}$ symmetric complex potentials have been found [27] which have a real discrete spectrum; the complex Morse potential is one such example.

Pseudo-Hermiticity and several related features such as pseudo-norm, pseudo-orthogonality, bi-orthogonal basis, etc, have been studied in detail [12]. It has been claimed that the $\mathcal{PT}$-symmetric potentials are pseudo-Hermitian ($H^{\#} = \boldsymbol{\eta} H^{\dagger} \boldsymbol{\eta}^{-1} = H$) when $\boldsymbol{\eta} = \mathcal{P}$. Furthermore, it has been discovered that several non-$\mathcal{PT}$-symmetric potentials such as the complex Morse potential are pseudo-Hermitian when $\boldsymbol{\eta}$ is taken to be an operator $\mathrm{e}^{-p\theta}$ [13] which effects an imaginary shift in the coordinate $\mathrm{e}^{p\theta} x\, \mathrm{e}^{-p\theta} = x + \mathrm{i}\theta$. Similarly, several other non-Hermitian Hamiltonians including H_{H-N} (see above), giving rise to a real discrete spectrum, have been shown to be pseudo-Hermitian under gauge-like transformation [28]. With the current renewed interest in real eigenvalues of non-Hermitian Hamiltonians via $\mathcal{PT}$-symmetry settling to the notion of (weak-) pseudo-Hermiticity [29], the occurrence of complex conjugate pairs of eigenvalues has also been emphasized and investigated.

2.2. *Pseudo-Hermiticity and pseudo-unitarity*

Let the metric $\boldsymbol{\eta}$ be a non-singular matrix, let $\mathbf{x}$ and $\mathbf{y}$ be two arbitrary vectors and let c be a constant scalar. Let O be an operator which will facilitate a transformation. Let us further define 'hash' (#) [6] which denotes distortion of the usual Hermitian adjoint operation †. We demand that

$$(c)^{\#} = c^* \tag{1}$$

$$(\mathbf{x})^{\#} = \mathbf{x}^{\dagger} \boldsymbol{\eta} \tag{2}$$

$$(\mathsf{O}\mathbf{x})^{\#} = \mathbf{x}^{\dagger} \boldsymbol{\eta} \mathsf{O}^{\#}. \tag{3}$$

We now define the (distorted) pseudo-inner product as $(\mathbf{x}|\mathbf{y})$. For familiarity, note that

$$(\mathbf{x}|\mathbf{y}) = \mathbf{x}^{\#}\mathbf{y} = \mathbf{x}^{\dagger}\boldsymbol{\eta}\mathbf{y} = \langle \mathbf{x}|\boldsymbol{\eta}\mathbf{y}\rangle \tag{4}$$

Let O transform $\mathbf{z}$ to $\mathbf{z}'$. We now insist that $(\mathbf{x}'|\mathbf{y}')$, or equivalently, $\langle \mathbf{x}|\boldsymbol{\eta}\mathbf{y}\rangle$ are left invariant. Thus,

$$(\mathbf{x}'|\mathbf{y}') = (\mathsf{O}\mathbf{x}|\mathsf{O}\mathbf{y}) = \mathbf{x}^{\dagger}\boldsymbol{\eta}\mathsf{O}^{\#}\mathsf{O}\mathbf{y}. \tag{5}$$

Alternatively, we can write

$$\langle \mathbf{x}'|\boldsymbol{\eta}\mathbf{y}'\rangle = \langle \mathsf{O}\mathbf{x}|\boldsymbol{\eta}\mathsf{O}\mathbf{y}\rangle = \mathbf{x}^{\dagger}\mathsf{O}^{\dagger}\boldsymbol{\eta}\mathsf{O}\mathbf{y} = x^{\dagger}\boldsymbol{\eta}\boldsymbol{\eta}^{-1}\mathsf{O}^{\dagger}\boldsymbol{\eta}\mathsf{O}\mathbf{y}. \tag{6}$$

By comparing equations (5) and (6), we obtain the definition of the 'hash' operation as

$$\mathsf{O}^{\#} = \boldsymbol{\eta}^{-1}\mathsf{O}^{\dagger}\boldsymbol{\eta}. \tag{7}$$

Now the required invariance can be achieved if, in equation (5), we set

$$\mathsf{O}^{\#}\mathsf{O} = 1 \Rightarrow \mathsf{O}^{\#} = \mathsf{O}^{-1}. \tag{8}$$

When $\boldsymbol{\eta} = 1$, 'hash' goes to 'dagger' and equation (4) defines unitarity. Thus, for any other non-singular $\boldsymbol{\eta}$, equations (7) and (8) define pseudo-unitarity.

Let us also check whether $\mathsf{O}^{\#}$ is $\boldsymbol{\eta}$-pseudo-adjoint. We note that

$$(\mathsf{O}^{\#}\mathbf{x}|\mathbf{y}) = \mathbf{x}^{\dagger}\boldsymbol{\eta}(\mathsf{O}^{\#})^{\#}\mathbf{y} = \mathbf{x}^{\dagger}\boldsymbol{\eta}\mathsf{O}\mathbf{y} = (\mathbf{x}|\mathsf{O}\mathbf{y}) \tag{9}$$

and we also note that

$$(\mathsf{O}^{\#})^{\#} = (\boldsymbol{\eta}^{-1}\mathsf{O}^{\dagger}\boldsymbol{\eta})^{\#} = \boldsymbol{\eta}^{-1}(\boldsymbol{\eta}^{-1}\mathsf{O}^{\dagger}\boldsymbol{\eta})^{\dagger}\boldsymbol{\eta} = \boldsymbol{\eta}^{-1}\boldsymbol{\eta}^{\dagger}\mathsf{O}\boldsymbol{\eta}^{-1^{\dagger}}\boldsymbol{\eta} = \mathsf{O} \qquad \text{iff} \quad \boldsymbol{\eta}^{\dagger} = \boldsymbol{\eta}. \tag{10}$$

Hence, it turns out that $\mathsf{O}^{\#}$ will be pseudo-adjoint if $(\mathsf{O}^{\#})^{\#} = \mathsf{O}$ and this requires that the metric should be Hermitian.

We know that any unitary matrix, U, can be written in Cayley's form, $\mathsf{U} = \mathrm{e}^{\mathrm{i}\mathsf{H}'}$, where H' is a Hermitian matrix. Similarly, let $\mathsf{O} = \mathrm{e}^{\mathrm{i}\mathsf{H}}$. For O to be pseudo-unitary, i.e., $\mathsf{O}^{\#} = \mathsf{O}^{-1}$, implies that

$$\begin{aligned}\boldsymbol{\eta}^{-1}\,\mathrm{e}^{-\mathrm{i}\mathsf{H}^{\dagger}}\boldsymbol{\eta} &= \sum_j \frac{(-\mathrm{i})^j\boldsymbol{\eta}^{-1}(\mathsf{H}^{\dagger})^j\boldsymbol{\eta}}{j!} \\ &= \sum_j \frac{(-\mathrm{i})^j(\boldsymbol{\eta}^{-1}\mathsf{H}^{\dagger}\boldsymbol{\eta})^j}{j!} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\eta}^{-1}\mathsf{H}^{\dagger}\boldsymbol{\eta}} = \mathrm{e}^{-\mathrm{i}\mathsf{H}},\end{aligned} \tag{11}$$

H should be pseudo-Hermitian, i.e., $\mathsf{H}^{\#} = \mathsf{H}$.

A given matrix can display pseudo-Hermiticity with respect to several metrics. We have recently proposed [30] a unique representation of $\boldsymbol{\eta}$ if the matrix can be diagonalized. Cayley's form, as adopted above, for a pseudo-unitary matrix reveals that the eigenvalues of a pseudo-unitary matrix are either unimodular-like, $\mathrm{e}^{\mathrm{i}\lambda}$ with real λ, or, they occur in pairs like e^{λ_1} and e^{λ_2} such that $|\lambda_1\lambda_2| = 1$. A non-unitary matrix possessing such an eigenvalue-structure will be pseudo-unitary.

Also, note that $(\mathsf{O}_1\mathsf{O}_2)^{\#} = \boldsymbol{\eta}^{-1}(\mathsf{O}_1\mathsf{O}_2)^{\dagger}\boldsymbol{\eta} = \boldsymbol{\eta}^{-1}\mathsf{O}_2{}^{\dagger}\boldsymbol{\eta}\boldsymbol{\eta}^{-1}\mathsf{O}_1\boldsymbol{\eta} = \mathsf{O}_2{}^{\#}\mathsf{O}_1{}^{\#}$. For other interesting results, see [6].

2.3. Pseudo-unitary group

We now prove that with respect to a fixed metric, $\boldsymbol{\eta}$, pseudo-unitary matrices form a group under matrix multiplication.

(i) Pseudo-unitary matrices are *closed* under multiplication. Let O_1 and O_2 be two arbitrary pseudo-unitaries. Then,

$$(\mathsf{O}_1\mathsf{O}_2)^\# = \boldsymbol{\eta}^{-1}(\mathsf{O}_1\mathsf{O}_2)^\dagger\boldsymbol{\eta} = \boldsymbol{\eta}^{-1}\mathsf{O}_2{}^\dagger\mathsf{O}_1{}^\dagger\boldsymbol{\eta}$$
$$= \boldsymbol{\eta}^{-1}\mathsf{O}_2{}^\dagger\boldsymbol{\eta}\boldsymbol{\eta}^{-1}\mathsf{O}_1{}^\dagger\boldsymbol{\eta} = \mathsf{O}_2{}^\#\mathsf{O}_1{}^\# = \mathsf{O}_2{}^{-1}\mathsf{O}_1{}^{-1} = (\mathsf{O}_1\mathsf{O}_2)^{-1}. \quad (12)$$

(ii) If O is pseudo-unitary under $\boldsymbol{\eta}$, then O^{-1} is also pseudo-unitary under the same metric. To see this,

$$(\mathsf{O}^{-1})^\# = \boldsymbol{\eta}^{-1}(\mathsf{O}^{-1})^\dagger\boldsymbol{\eta} = \boldsymbol{\eta}^{-1}(\mathsf{O}^\dagger)^{-1}\boldsymbol{\eta}$$
$$= (\boldsymbol{\eta}^{-1}\mathsf{O}^\dagger\boldsymbol{\eta})^{-1} = (\mathsf{O}^\#)^{-1} = (\mathsf{O}^{-1})^{-1} = \mathsf{O}. \quad (13)$$

(iii) The identity matrix would act as the unit element of this symmetry transformation group.
(iv) The associativity of the product of pseudo-unitary matrices follows trivially.

This group, introduced in [31], is called $PU(N)$. An explicit construction for $PU(2)$ is given in that work, along with the generators of the group.

2.4. Pseudo-Hermitian matrices

A general 2 × 2 matrix has four complex entries, and hence eight real parameters. For having only real eigenvalues, first of all, the diagonal elements should either be real or a complex-conjugate pair. Furthermore, let us write a general matrix

$$\mathsf{H} = \begin{bmatrix} z_1 & a_2 + \mathrm{i}b_2 \\ a_3 + \mathrm{i}b_3 & z_2 \end{bmatrix} \quad (14)$$

where z_1 and z_2 have the restrictions mentioned above. For eigenvalues to be real or a complex-conjugate pair, there appears a further restriction: $b_2a_3 + b_3a_2 = 0$. This then leads to four independent parameters for statistical independence among the matrix elements. Thus, in random matrices considered here, there will only be a maximum of four statistically independent parameters. Note that a subset of these matrices will be Hermitian where $a_2 = a_3$, $b_2 = -b_3$, and where z_1, z_2 are real.

In the following, we consider three-parameter and four-parameter cases as they give distinct results for spacing distributions. Certain two-parameter cases will also have spacing distributions similar to these cases.

A pseudo-Hermitian matrix can be diagonalized by a pseudo-unitary matrix (see equations (18), (28), (32), (40) and (46)). In the same vein, a pseudo-unitary transformation defines a class of pseudo-Hermitian matrices (see equation (17)).

3. Pseudo-unitary invariant ensembles

Given a matrix H, the matrix elements will be drawn from a Gaussian distribution [32]

$$P(\mathsf{H}) = \mathcal{N}\exp\left(-\frac{\mathrm{tr}(\mathsf{H}\,\mathsf{H}^\dagger)}{2\sigma^2}\right) \quad (15)$$

where $\mathcal{N}$ is the normalization constant. We demand that this matrix is pseudo-Hermitian with respect to a metric $\boldsymbol{\delta}$. The specific distribution is chosen for convenience, as in [33]. Furthermore, there are two possibilities, for the metric may lead to an indefinite norm or a definite norm. We present each of these cases separately.

3.1. Random matrices with three independent parameters

As discussed in section 2.4, there are several cases. We now demonstrate that different forms of matrices along with the nature of the pseudo-norm or the metric lead to different spectral fluctuations. Since this aspect is novel, it is important to illustrate with disparate cases and classes.

3.1.1. Cases with an indefinite norm

Example 1. We now consider a Hamiltonian H with a simple matrix representation where diagonal elements are real and equal and off-diagonal elements are purely imaginary, given by [31]

$$\mathsf{H} = \{\mathsf{H}_{ij}\} = \begin{bmatrix} a & -\mathrm{i}b \\ \mathrm{i}c & a \end{bmatrix} \tag{16}$$

which can be diagonalized by D, i.e.

$$\mathsf{H} = \mathsf{D}\begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix}\mathsf{D}^{-1}. \tag{17}$$

The eigenvalues of H are $a \pm \sqrt{bc}$ $(bc > 0)$. H is pseudo-Hermitian with respect to the metric given by the Pauli matrix, σ_y.

The corresponding matrix, D,

$$\mathsf{D} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{i}/r \\ \mathrm{i}r & 1 \end{bmatrix} \tag{18}$$

is pseudo-unitary under the metric

$$\boldsymbol{\eta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{19}$$

The pseudo-norm (see equation (4)), $\langle \mathbf{x}|\boldsymbol{\eta}\mathbf{x}\rangle$ is $2\Re(x_1^* x_2)$, with $\mathbf{x}$ as a column vector with x_1 and x_2 as components, and is indefinite.

The eigenvalues are

$$E_\pm = a \pm \left[\frac{c}{2r} + \frac{br}{2}\right] \tag{20}$$

where $r = \sqrt{c/b}$ $(0 \leqslant r \leqslant \infty)$.

According to equation (15), the joint probability distribution of a, b, c is

$$P(a, b, c) = \frac{1}{2(\pi\sigma^2)^{\frac{3}{2}}}\,\mathrm{e}^{-\frac{1}{2\sigma^2}[2a^2+b^2+c^2]}. \tag{21}$$

From equations (5) and (20), we have the following relations:

$$a = \frac{E_+ + E_-}{2} \qquad b = \frac{E_+ - E_-}{2r} \qquad c = \frac{r(E_+ - E_-)}{2}. \tag{22}$$

The Jacobian, J, connecting (a, b, c) and (E_+, E_-, r) is $\frac{|E_+ - E_-|}{2r}$. With these, the joint probability distribution function (j.p.d.f.) of eigenvalues is

$$P(E_+, E_-) = \frac{|E_+ - E_-|}{2(\pi\sigma^2)^{\frac{3}{2}}} K_0\left(\frac{(E_+ - E_-)^2}{4\sigma^2}\right)\mathrm{e}^{-\frac{(E_+ + E_-)^2}{4\sigma^2}}. \tag{23}$$

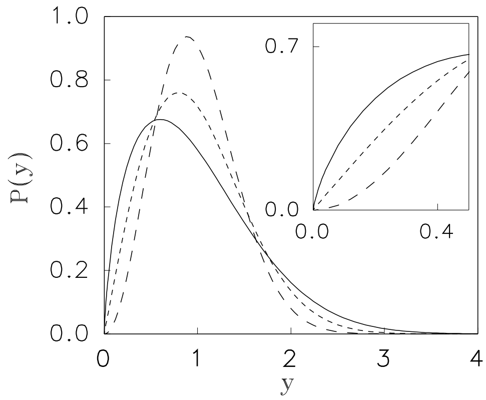

Figure 1. The nearest-neighbour level-spacing distribution (25) (full curve) is plotted here for systems where the matrices have three independent parameters and the pseudo-norm is indefinite. A comparison is made with the spacing distributions for the orthogonal (small-dashed curve) and unitary ensembles (medium-dashed curve) which are time-reversal invariant and non-invariant, respectively. The inset shows that the level repulsion is different from linear or quadratic; it is $S \log \frac{1}{S}$ for this case when parity is also broken along with time reversal. In all the figures, we take $\sigma = 1$.

Integrating with respect to E_- gives the average density. The shape of the level density resembles that for Wigner–Dyson ensembles [31]. This is not yet amenable to an analytically closed form.

The nearest-neighbour level-spacing distribution, $P(S)$, is given in terms of the j.p.d.f. by [32]

$$
\begin{aligned}
P(S) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(E_+, E_-)\delta(S - |E_+ - E_-|)\,\mathrm{d}E_+\,\mathrm{d}E_- \\
&= \frac{S}{\pi\sigma^2} K_0\left(\frac{S^2}{4\sigma^2}\right).
\end{aligned} \tag{24}
$$

In terms of the mean-level spacing, D, the normalized spacing distribution assumes the following form

$$
P(y) = \frac{\Gamma^4\left(-\frac{1}{4}\right)}{32\pi^3} y K_0\left(\frac{2\Gamma^4\left(\frac{3}{4}\right)}{\pi^2} y^2\right) \tag{25}
$$

where $y = S/D$.

This result is very interesting (figure 1), particularly for its behaviour near zero spacing. Near $y = 0$, the probability distribution varies as $y \log \frac{1}{y}$. This follows from the asymptotic properties of the modified Bessel function of the second kind.

Remarkably, for several other examples, the nearest-neighbour level-spacing distribution turns out to be equation (25). It should be noted that, in the case discussed above, the metric under which H and D are invariant is different. We present one example where H and D are invariant under the same metric below.

Example 2. We have now a form with three parameters

$$
\mathsf{H} = \begin{bmatrix} a+c & \mathrm{i}b \\ \mathrm{i}b & a-c \end{bmatrix} \tag{26}
$$

which is pseudo-Hermitian with respect to $\boldsymbol{\zeta}$:

$$
\boldsymbol{\zeta} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{27}
$$

The eigenvalues are $a \pm \sqrt{c^2 - b^2}$.

Note that the diagonalizing matrix, D, is

$$\mathsf{D} = \frac{1}{\sqrt{\cos 2\theta}} \begin{bmatrix} \cos\theta & \mathrm{i}\sin\theta \\ -\mathrm{i}\sin\theta & \cos\theta \end{bmatrix}. \tag{28}$$

As a result, the following relations hold

$$a = \frac{E_+ + E_-}{2} \qquad b = \frac{E_- - E_+}{2}\tan 2\theta \qquad c = \frac{E_+ - E_-}{2}\sec 2\theta \tag{29}$$

where $\theta = -\frac{1}{2}\sin^{-1}(b/c)\left(\frac{-\pi}{4} < \theta < \frac{\pi}{4}\right)$. The Jacobian is $\left|-\frac{E_+ - E_-}{2}\sec 2\theta\right|$.

Notice that D is pseudo-unitary. This can be verified by the non-unitarity along with the nature of the eigenvalues, as stated in section 2.2. Whenever the metric is diagonal with positive elements, the pseudo-norm is positive definite (see section 3.1.2, for example). In contrast, any other metric leads to an indefinite pseudo-norm. The spacing distribution turns out to be that in equation (25). However, as discussed in section 2.4, the condition leading to real eigenvalues for an arbitrary non-Hermitian matrix separates various possible classes in terms of the diagonal elements being real or a complex-conjugate pair, and so on. As mentioned above, the result for the spacing distribution for *all* these classes turns out to be equation (25).

3.1.2. A case with a definite norm. Present understanding suggests that the definiteness or indefiniteness of the metric leads to the unconditional or a conditional existence of real eigenvalues. We have the following pseudo-Hermitian matrix to illustrate the case with a definite norm

$$\mathsf{H} = \{\mathsf{H}_{ij}\} = \begin{bmatrix} a & -\mathrm{i}\epsilon c \\ \mathrm{i}c/\epsilon & b \end{bmatrix} \tag{30}$$

where a, b, c are real and ϵ is a fixed parameter. Note that this matrix becomes Hermitian as $\epsilon = 1$, thus it is a parametric distortion from Hermiticity.

The eigenvalues of H are

$$E_\pm = \tfrac{1}{2}\left[(a+b) \pm \sqrt{(a-b)^2 + 4c^2}\right]. \tag{31}$$

Due to the fact that the eigenvalues are real unconditionally, we expect the spacing distribution to be different from equation (25).

H can be diagonalized by D where

$$\mathsf{D} = \begin{bmatrix} \cos\theta & \mathrm{i}\epsilon\sin\theta \\ \mathrm{i}\sin\theta/\epsilon & \cos\theta \end{bmatrix} \tag{32}$$

is pseudo-unitary with respect to the metric

$$\boldsymbol{\eta} = \begin{bmatrix} 1/\epsilon & 0 \\ 0 & \epsilon \end{bmatrix}. \tag{33}$$

Note that H is pseudo-Hermitian, also with respect to $\boldsymbol{\eta}$.

Note that the eigenvalues here are unconditionally real, unlike the examples in section 3.1.1. The pseudo-norm (defined in equation (4)), $\langle \mathbf{x}|\boldsymbol{\eta}\mathbf{x}\rangle$, is $\epsilon x_1^2 + \frac{x_2^2}{\epsilon}$ with x_1 and x_2 being the components, and is positive-definite for real ϵ. Since $\boldsymbol{\eta}$ is expressible [34] as $\mathsf{O}\mathsf{O}^\dagger$, H admits only real eigenvalues, equation (31).

Accordingly, the j.p.d.f. of a, b, c is

$$P(a, b, c) = \mathcal{N}\exp[-(a^2 + b^2 + 2\cosh(2\gamma)c^2)] \tag{34}$$

where $\epsilon = e^\gamma$ (this choice makes the norm positive-definite) and $\mathcal{N}$ is a normalization constant.

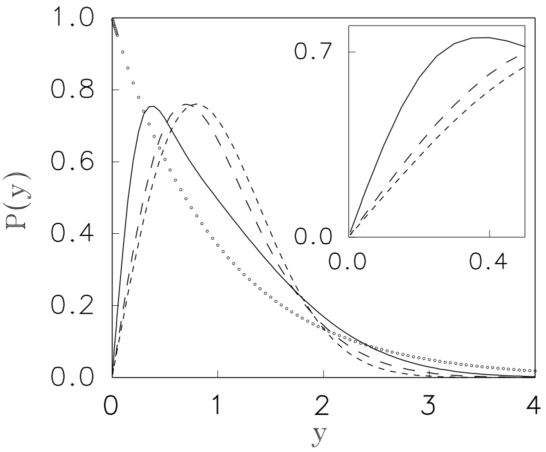

Figure 2. The nearest-neighbour level-spacing distribution (37) is plotted here for systems where the matrices have three independent parameters and the pseudo-norm is definite. There is a parameter, γ, which is taken as 0 (small-dashed curve), 1 (medium-dashed curve), and 2 (full curve) to show different possibilities. $\gamma = 0$ corresponds to the GOE result. Also, the Poisson statistics are plotted (open circles).

We have the following relations

$$\begin{aligned} a &= \tfrac{1}{2}[(E_+ + E_-) + (E_+ - E_-)\cos 2\theta] \\ b &= \tfrac{1}{2}[(E_+ + E_-) - (E_+ - E_-)\cos 2\theta] \\ c &= \frac{E_+ - E_-}{2}\sin 2\theta \end{aligned} \tag{35}$$

with $\theta = \frac{1}{2}\tan^{-1}\frac{2c}{a-b}$ $(-\pi/4 < \theta < \pi/4)$. The Jacobian, J, connecting (a, b, c) and (E_+, E_-, θ) is $|E_+ - E_-|$. With these, the j.p.d.f. of the eigenvalues is obtained by integrating $P(E_+, E_-, \theta)$ over θ. We then obtain in terms of $S = E_+ - E_-$ and $t = (E_+ + E_-)/2$:

$$P(S, t; \gamma) \sim S\,\mathrm{e}^{-\frac{t^2}{2\sigma^2} - \frac{S^2\cosh^2\gamma}{4\sigma^2}} I_0\left[\frac{S^2}{4\sigma^2}\sinh^2\gamma\right]. \tag{36}$$

With this, we find the normalized nearest-neighbour level-spacing distribution in terms of $y = S/D$

$$P(y) = 2\mathcal{A}\sqrt{1 - \tanh^4\gamma}\,y\,\mathrm{e}^{-\mathcal{A}y^2} I_0(\mathcal{A}y^2\tanh^2\gamma) \tag{37}$$

where the constant $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{\pi}{4}\frac{1 + \tanh^2\gamma}{\cosh^2\gamma}\,{}_2F_1^2\left(\frac{3}{4}, \frac{5}{4}, 1, \tanh^4\gamma\right) \tag{38}$$

in terms of a hypergeometric function.

This has the interesting feature of level repulsion as seen in figure 2, and is quite distinctive due to the fact that the pseudo-norm is definite. The degree of level repulsion is 1 as in the case of the Gaussian orthogonal ensemble (GOE). However, as a function of γ, it varies as seen in figure 2. This spacing distribution also appears in another recent work on a non-invariant Gaussian ensemble of 2 × 2 orthogonal matrices [35].

For $\gamma = 0$, the matrix H becomes Hermitian, and the spacing distribution becomes that for random Hermitian matrices with three parameters, which agrees with the GOE [36].

The statistics for various real values of γ (= 0, 1, 2) are displayed in figure 2. Thus, this result gives the spacing distribution as we deviate from Hermiticity (and the GOE as well).

3.2. Random matrices with four independent parameters

In the first of the two cases considered in this section, the diagonal elements are a complex-conjugate pair whereas, in the other case, they are real. The second case corresponds to a Hamiltonian matrix introduced by Sudarshan [1].

Case 1. The matrix we consider is

$$\mathsf{H} = \begin{bmatrix} a + \mathrm{i}b & c \\ d & a - \mathrm{i}b \end{bmatrix} \tag{39}$$

where a, b, c and d are real parameters. This matrix is pseudo-Hermitian with respect to $\boldsymbol{\eta}$ of equation (19). The eigenvalues $E_\pm = a \pm \sqrt{cd - b^2}$ are real if $cd > b^2$.

The diagonalizing matrix is found to be

$$\mathsf{D} = \begin{bmatrix} r\,\mathrm{e}^{\mathrm{i}\theta}/\sin\theta & -r\,\mathrm{e}^{-\mathrm{i}\theta}/\sin\theta \\ 1 & 1 \end{bmatrix}. \tag{40}$$

D is not pseudo-unitary with respect to $\boldsymbol{\eta}$. The eigenvalues of D satisfy the condition, $|\lambda_1\lambda_2| = 1$, and we expect it to be pseudo-unitary with respect to some indefinite metric.

According to equation (17), the following relations can be easily verified

$$a = \frac{E_+ + E_-}{2} \qquad b = \frac{E_+ - E_-}{2}\tan\theta$$
$$c = \frac{r(E_+ - E_-)}{\sin 2\theta} \qquad d = \frac{(E_+ - E_-)\tan\theta}{2r} \tag{41}$$

where $r = \frac{b}{d}$, $\cot\theta = \sqrt{\frac{cd}{b^2} - 1}$. The parameters b, c, d have to be such that $\cot\theta > 0$. For this, b, c, d are required to be of the same sign. Moreover, for the distributions to remain well-defined, r must be positive. We are thus restricted to having b, c, d being all positive semi-definite. The algebraic relations among the parameters lead to restrictions in the integration domains. Thus, $\theta \in (0, \pi/2)$, and $r \in (0, \infty)$. The Jacobian is $|-S^2\sec^2\theta/4r|$, where we denote $(E_+ + E_-)/2$ and $(E_+ - E_-)$ by t and S, respectively.

The j.p.d.f. of the eigenvalues in terms of t and S is given by

$$P(t, S) \sim S\,\mathrm{e}^{-\frac{t^2}{\sigma^2} + \frac{S^2}{4\sigma^2}}\,\mathrm{erfc}\left(\frac{S}{\sqrt{2}\sigma}\right) \tag{42}$$

where erfc(x) is the complementary error function.

Finally, for this four-parameter matrix, the normalized spacing distribution has a novel form as a function of the scaled variable, $y = S/D$ as above

$$P(y) = \frac{\mathcal{B}^2}{2(\sqrt{2} - 1)}\,y\,\mathrm{e}^{\frac{\mathcal{B}^2 y^2}{4}}\,\mathrm{erfc}\left(\frac{\mathcal{B}y}{\sqrt{2}}\right) \tag{43}$$

where

$$\mathcal{B} = \frac{2(\sqrt{2} - \log(1 + \sqrt{2}))}{\sqrt{2} - 1}. \tag{44}$$

This shows a linear level repulsion (figure 3) with a large slope, in distinction with the Wigner surmise for time-reversal invariant ensembles. It has a completely different behaviour in its details. Along with the results of section 3.1 and the next subsection, equation (43) exemplifies the richness of the pseudo-Hermitian ensembles.

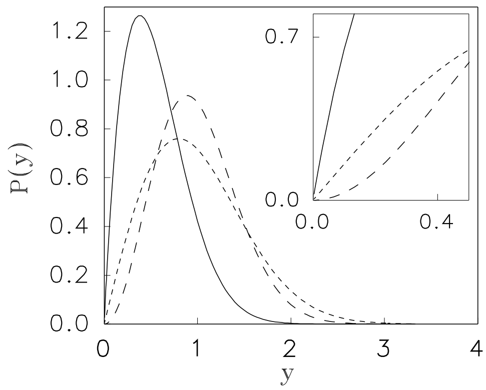

Figure 3. The nearest-neighbour level-spacing distribution (43) (full curve) is plotted here for the systems where the matrices have four independent parameters and the pseudo-norm is indefinite. A comparison is made with the spacing distributions for the orthogonal (small-dashed curve) and unitary ensembles (medium-dashed curve) which are time-reversal invariant and non-invariant, respectively. The inset shows that the level repulsion varies near the origin as αy (with α being the pre-factor in equation (43)) for this case when parity is also broken along with time reversal.

Case 2. The matrix we consider is

$$\mathsf{H} = \begin{bmatrix} a+b & d+\mathrm{i}c \\ -d+\mathrm{i}c & a-b \end{bmatrix} \tag{45}$$

where a, b, c and d are real parameters. The eigenvalues $E_\pm = a \pm \sqrt{b^2 - c^2 - d^2}$ are real if $b^2 > c^2 + d^2$.

The diagonalizing matrix is found to be

$$\mathsf{D} = \begin{bmatrix} \mathrm{i}\cos\theta & \mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{-\mathrm{i}\phi}\sin\theta & -\mathrm{i}\cos\theta \end{bmatrix}. \tag{46}$$

The matrix H is pseudo-Hermitian with respect to $\boldsymbol{\zeta}$ in equation (27). Also, D is pseudo-unitary with respect to the same metric, $\boldsymbol{\zeta}$.

According to equation (17), the following relations can be easily verified

$$\begin{aligned} a &= \frac{E_+ + E_-}{2} \qquad & b &= \frac{(E_+ - E_-)}{2}\sec 2\theta \\ c &= -\frac{(E_+ - E_-)}{2}\tan 2\theta\cos\phi \qquad & d &= \frac{(E_+ - E_-)}{2}\tan 2\theta \sin\phi \end{aligned} \tag{47}$$

where $\phi = -\tan^{-1}\frac{c}{d}$, $\theta = \frac{1}{2}\sin^{-1}\sqrt{\frac{c^2+d^2}{b^2}}$ $\left(0 < \theta < \frac{\pi}{2}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$. The Jacobian is $\frac{S^2}{4}\sec 2\theta \tan 2\theta$.

The j.p.d.f. of the eigenvalues in terms of t and S is again given by equation (42). The normalized spacing distribution is also exactly as in equation (43). This exhibits the strong generality of the result.

For these most general pseudo-Hermitian matrices, the level repulsion is linear near zero spacing (figures 3 and 4).

4. Summary and concluding remarks

In this paper, quantum mechanics with an indefinite metric is shown to be connected with $\mathcal{PT}$-symmetry or, more generally, with pseudo-Hermiticity. This leads us to conclude that systems

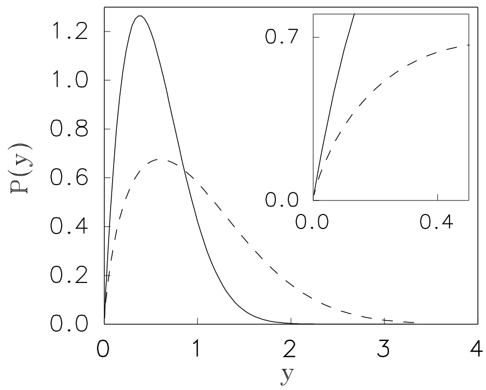

Figure 4. This exhibits the rich structure of the generality ('universality') of spectral fluctuations as a comparison is made among the main results for the spacing distribution. The full and dashed curves correspond to equations (43) and (25), respectively, showing linear and non-algebraic level repulsion, respectively.

where $\mathcal{P}$ and $\mathcal{T}$ are broken find their description in quantum mechanics with an indefinite metric. The connection of this with physical situations of great importance has already been discussed in section 1 and in [31].

In a simple manner, the pseudo-unitary group is constructed here. This is in a logical scheme from a quantum to a random matrix description. Perhaps the most well-studied characterization is the nearest-neighbour level-spacing distribution, $P(S)$. This gives the frequency with which a certain spacing between adjacent levels occurs [32]. For the Wigner–Dyson ensembles, $P(S) \sim S^{\beta_0}\,\mathrm{e}^{-\gamma S^2}$ where β_0 is 1, 2 and 4 for the orthogonal, unitary and symplectic ensembles. A wide variety of systems display universal properties possessed by random matrix ensembles, as can be seen in [32, 37, 39]. However, there are systems that display intermediate statistics [40–42]. These systems range from examples of billiards in polygonal enclosures, three-dimensional Anderson model at the metal–insulator transition point, and others. On the other hand, there have long since been important developments on non-Hermitian ensembles where the eigenvalues are complex [32, 38, 39], and where an ensemble of unstable states is considered [43]. Clearly, the ensemble developed here does not fall into any of the known categories and, indeed, displays some novel features as shown here.

It is important to note that the eigenvalues of a pseudo-Hermitian matrix are real only under certain conditions on the parameters giving the matrix elements. These conditions restrict the domain of integration over the pseudo-unitary group. This, in turn, leads to restrictions on the parameters defining the pseudo-unitary transformation (e.g. θ and ϕ in equation (46)).

The central results are the new spacing distributions given by equations (25) and (43). For the most general 2×2 matrix, there are four independent parameters, and the spacing distribution is given by equation (43). This is whether or not H and D are respectively pseudo-Hermitian and pseudo-unitary with respect to the same metric. Similarly, for the general case with three independent parameters, the spacing distribution is equation (25). As the spacing becomes small, the degree of level repulsion is known. The asymptotic behaviour near y equal to zero is as follows:

$$\begin{aligned} &P(y) \to \alpha y && \text{as} \quad y \to 0 \text{ (three independent parameters)} \\ &P(y) \to y \log \tfrac{1}{y} && \text{as} \quad y \to 0 \text{ (four independent parameters).} \end{aligned} \tag{48}$$

The quantity α is much larger than for the GOE, in addition to differences in details.

We have also presented a case where, by variation of a parameter, the departure from Hermiticity occurs and spacing distribution is given by equation (37). For this case, the behaviour near $y = 0$ is $yf(\gamma)$. The level repulsion remains linear as in the GOE, but $f(\gamma)$ gives the slope.

The scenario emerging from this work is the following. When a quantum system violates time-reversal invariance, it is well known that the degree of level repulsion is two. In addition, if parity is broken, the degree of level repulsion becomes one, but with an entirely different slope. This can be a test for classifying systems where parity and time reversal both are preserved or violated. This scenario is when there are four statistically independent parameters, which is the most general case. Furthermore, for a special case when there remain three independent parameters, the level repulsion becomes non-algebraic ($S \log \frac{1}{S}$) if the pseudo-norm continues to be indefinite, and it remains linear if the pseudo-norm becomes definite.

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